Spaces of matrices with rank bounded above

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Notation

- $\mathbb{K}$ arbitrary field (possibly finite), n and p positive integers with $n \geq p$;
- $\mathrm{M}_{n,p}(\mathbb{K})$ the vector space of $n \times p$ matrices, entries in $\mathbb{K}$;
- $\mathrm{M}_n(\mathbb{K}) := \mathrm{M}_{n,n}(\mathbb{K})$.

Linear subspaces V and W of $\mathrm{M}_{n,p}(\mathbb{K})$ are **equivalent** when

$$\exists (P,Q) \in \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_p(\mathbb{K}) : \quad W = PVQ.$$

The **upper-rank** of a $V \subset \mathrm{M}_{n,p}(\mathbb{K})$ is

$$\overline{\mathrm{rk}}(V) := \max\{\mathrm{rk}\, M \mid M \in V\}.$$

Main problem

Given r s.t. $0 < r < p$, classify the linear subspaces of $M_{n,p}(\mathbb{K})$ with upper rank r (up to equivalence).
Solution known for small r (Schur: $r = 1$; Atkinson: $2 \leq r \leq 3$).
For general values, no fully encompassing answer is known. In particular, the maximal spaces are not known.
More realistic goals:

(i) Find the maximal dimension for a subspace with upper rank r.

(ii) Classify those subspaces with a dimension "close to" the maximal one.

Flanders's theorem

Theorem (Flanders, 1962)

Let $r \in [\![1, p-1]\!]$ *and* V *a linear subspace of* $\mathrm{M}_n(\mathbb{K})$ *with* $\overline{\mathrm{rk}}(V) = r$. *Then:*

(a) $\dim V \leq rn$;

(b) *If* $\dim V = rn$ *and* $n > p$, *then*

$$V \sim C_{n,p,r} := \left\{ \begin{bmatrix} [?]_{n\times r} & [0]_{n\times(p-r)} \end{bmatrix} \right\}.$$

(c) *If* $\dim V = rn$ *and* $n = p$, *then*

$$V \text{ or } V^T \sim \left\{ \begin{bmatrix} [?]_{n\times r} & [0]_{n\times(n-r)} \end{bmatrix} \right\}.$$

Bibliography on Flanders's theorem

The case $\#\mathbb{K} > r$:

H. Flanders, *On spaces of linear transformations with bounded rank.* J. Lond. Math. Soc. **37** (1962) 10–16.

The case $n = p$ and $r = n - 1$ for an arbitrary field:

J. Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables.* Arch. Math. **1** (1949) 282–287.

Arbitrary field, arbitrary n, p, r:

R. Meshulam, *On the maximal rank in a subspace of matrices.* Quart. J. Math. Oxford (2) **36** (1985) 225–229.

The more general case extended to affine subspaces:

C. de Seguins Pazzis, *The affine preservers of non-singular matrices.* Arch. Math. **95** (2010) 333–342.

The naive conjecture:
A subspace V with upper-rank r may be extended to one with the maximal dimension nr.

Equivalently, if $n > p$, then the matrices of V vanish on some common $(p-r)$-dimensional subspace of $\mathbb{K}^p$ (if $n = p$, this should hold for V or for V^T).

True for $r = 1$ (Schur)! False in general!

Basic counterexample: Given $(s,t) \in [\![0,n]\!] \times [\![0,p]\!]$,

$$\mathcal{R}(s,t) := \left\{ \begin{bmatrix} [?]_{s\times t} & [?]_{s\times(p-t)} \\ [?]_{(n-s)\times t} & [0]_{(n-s)\times(p-t)} \end{bmatrix} \right\} \subset \mathrm{M}_{n,p}(\mathbb{K}).$$

Note : $\overline{\mathrm{rk}}\big(\mathcal{R}(s,t)\big) \leq s+t$, with equality whenever $s+t \leq p$.
Note : $C_{n,p,r} = \mathcal{R}(0,r)$.

Another example:
If n odd, the space $\mathrm{A}_n(\mathbb{K})$ of all alternating matrices is a maximal subspace with upper-rank $n-1$.

Atkinson and Lloyd: can we classify spaces $V \subset \mathrm{M}_n(\mathbb{K})$ with $\overline{\mathrm{rk}}(V) = r$ and

$$nr - (r-1) \leq \dim V \quad ?$$

Theorem (Atkinson and Lloyd, 1980)

Let $r \in [\![1, n-1]\!]$ and V maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with $\overline{\mathrm{rk}}(V) = r$.
Assume $\#\mathbb{K} > r$ and $\dim V \geq nr - r + 1$.
Then:

(a) *Either $\dim V = nr$;*

(b) *Or $V \sim \mathcal{R}(1, r-1)$, or $V \sim \mathcal{R}(r-1, 1)$.*

Bibliography on Atkinson and Lloyd's theorem

The original result:

M.D. Atkinson, S. Lloyd, *Large spaces of matrices of bounded rank.* Quart. J. Math. Oxford (2). **31** (1980) 253–262.

Extension to $n > p$:

L.B. Beasley, *Null spaces of spaces of matrices of bounded rank.* Current Trends in Matrix Theory, 45–50, Elsevier, 1987.

What about "small" finite fields? Counter-example for $\mathbb{F}_2$: the space

$$\left\{ \begin{bmatrix} a & c & d \\ 0 & b & e \\ 0 & 0 & a+b \end{bmatrix} \mid a, b, c, d, e \in \mathbb{F}_2 \right\}.$$

Recent results

No significant new result until:

Theorem (de Seguins Pazzis, 2010)

Atkinson and Lloyd's theorem holds for an arbitrary field, except when $n = 3$, $r = 2$, $\#\mathbb{K} = 2$ and $\dim V = 5$, in which case the above counter-example is the only exception up to equivalence.

C. de Seguins Pazzis, The classification of large spaces of matrices with bounded rank, in press at Israel Journal of Mathematics, arXiv: http://arxiv.org/abs/1004.0298

Beasley's results are also successfully extended to an arbitrary field (no exception!).

Setting the proof up

Let V linear subspace of $M_n(\mathbb{K})$ with $\overline{\text{rk}}(V) = r \in [\![1, n-1]\!]$.
With no loss of generality, V contains

$$\begin{bmatrix} P & [?]_{r\times(n-r)} \\ [?]_{(n-r)\times r} & [?]_{(n-r)\times(n-r)} \end{bmatrix}$$

for some *invertible* $P \in \text{GL}_r(\mathbb{K})$.

Split every matrix of V with the same block sizes:

$$M = \begin{bmatrix} A(M) & C(M) \\ B(M) & D(M) \end{bmatrix}.$$

Then,

$$\det(A(M))\, D(M) = B(M)\,\widetilde{A(M)}\,C(M). \qquad (1)$$

(notation: $\widetilde{N} = \text{com}(N)^T$).

The decomposition

We consider a tower of linear subspaces of V:

$$V_3 \subset V_2 \subset V_1 \subset V.$$

where

$$\begin{cases} V_1 & := \operatorname{Ker} A \\ V_2 & := \operatorname{Ker} A \cap \operatorname{Ker} B \\ V_3 & = \operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C, \end{cases}$$

$$\dim V = \dim A(V) + \dim B(V_1) + \dim C(V_2) + \dim V_3.$$

Let

$$M_0 = \begin{bmatrix} P & C(M_0) \\ B(M_0) & D(M_0) \end{bmatrix} \in V \quad \text{and} \quad M = \begin{bmatrix} [0] & C(M) \\ B(M) & D(M) \end{bmatrix} \in V_1.$$

Key identities

Hence,

$$\det(P)\,D(M_0+M)=B(M_0+M)\,\widetilde{P}\,C(M_0+M). \tag{2}$$

Subtracting the identity for $M=0$ yields:

$$B(M)\,\widetilde{P}\,C(M)=D(M)-B(M)\,\widetilde{P}\,C(M_0)-B(M_0)\,\widetilde{P}\,C(M).$$

In particular $V_3=\{0\}$.

Next, one computes the polar form
$(M,N)\mapsto f(M+N)-f(M)-f(N)$ on both sides :

$$\forall M,N\in V_1,\quad B(M)\,\widetilde{P}\,C(N)+B(N)\,\widetilde{P}\,C(M)=0.$$

In particular, for every $(M,N)\in V_1\times V_2$:

$$B(M)\,\mathrm{com}(P)^T c(N)=0.$$

Concluding the proof of inequality

In particular, with P invertible one gets

$$\forall M \in V_1,\ \forall N \in V_2, \quad B(M)P^{-1}C(N) = 0$$

leading to

$$\dim B(V_1) + \dim C(V_2) \leq r(n-r).$$

This yields

$$\dim V \leq r^2 + r(n-r) = nr.$$

To analyze the case $\dim V = nr$, the proof goes on using the above formulae ...

Sketch of proof of Atkinson-Lloyd's theorem: Step 1

We continue with the assumptions of the latest section. We now assume $\dim V > nr - r + 1$.
We show that either there is no non-zero matrix in V of the form

$$\begin{bmatrix} [0]_{r \times n} \\ [?]_{(n-r) \times n} \end{bmatrix}$$

or there is no non-zero matrix in V of the form

$$\begin{bmatrix} [0]_{n \times r} & [?]_{n \times (n-r)} \end{bmatrix}.$$

Using the above inequalities, we find

$$\operatorname{codim} A(V) < r-1.$$

We still have

$$\forall P \in A(V),\ \forall M \in V_1, \forall N \in V_2,$$

$$B(M)\,\widetilde{P}\,C(N) = 0.$$

Lemma

One has $B(V_1)=\{0\}$ or $C(V_2)=\{0\}$.

Proof.

Assume the contrary.

- $\text{span}\{\widetilde{P} \mid P \in A(V)\}$ does not act transitively on $\mathbb{K}^r$.
- Without loss of generality,

$$\forall P \in A(V),\ \text{com}(P)_{r,r} = 0.$$

- Contradiction with Flanders's theorem!!

□

If $B(V_1) = \{0\}$, then for $W := V^T$, $C(W_2) \subset B(V_1)^T = \{0\}$. Without loss of generality, $C(V_2) = \{0\}$. Then, $V_2 = \{0\}$.

Step 2: Looking for the commun null space

We now assume $V_2 = \{0\}$. In that case, there is a linear subspace $W \subset \mathrm{M}_{n,r}(\mathbb{K})$ and a linear map $\varphi : W \to \mathrm{M}_{n,n-r}(\mathbb{K})$ s.t.

$$V = \left\{ \begin{bmatrix} N & \varphi(N) \end{bmatrix} \mid N \in W \right\}.$$

Note that $\operatorname{codim} W < r - 1$.

Two lemmas

Lemma

One has

$$\forall N \in W, \ \operatorname{Im} \varphi(N) \subset \operatorname{Im}(N).$$

Lemma

There exists $C \in \mathrm{M}_{r,n-r}(\mathbb{K})$ *s.t.* $\forall N \in W, \ \varphi(N) = NC.$

Using the second lemma, take $P := \begin{bmatrix} I_r & -C \\ [0]_{(n-r)\times r} & I_{n-r} \end{bmatrix}$ and check that any matrix of VP has all last $n-r$ columns equal to zero.

Proof of the first lemma

Denote by U the space of matrices of W with last row 0. For $N \in U$, write

$$\begin{bmatrix} N & \varphi(N) \end{bmatrix} = \begin{bmatrix} K(N) & [?]_{(n-1)\times(n-r)} \\ [0]_{1\times r} & \varphi_n(K(N)). \end{bmatrix}$$

Note that:

- → $\operatorname{codim} K(U) < r - 1 < n - 2$;
- → if $\operatorname{rk} K(N) = r$, then $\varphi_n(K(N)) = 0$.
- → $\varphi_n : K(U) \to \mathrm{M}_{1,n-r}(\mathbb{K})$ is linear.

Proof of the first lemma (continued)

We deduce that $\varphi_n = 0$ by using a corollary of Flanders's theorem for affine subspaces:

Corollary

Let V' linear subspace of $M_{n,p}(\mathbb{K})$, with $n \geq p \geq r$ and $\dim V' > rn$. Then V' is spanned by its matrices of rank $> r$, unless $n = p = 2$, $r = 1$ and $\# \mathbb{K} = 2$.

Proof.

Assuming the contrary, take a linear hyperplane H of V' which contains every rank $> r$ matrix of V', and $H' := a + H$ with $a \in V' \setminus H$. Then, $\dim H' \geq rn$, $\overline{\text{rk}} H' \leq r$. □

Proof of the first lemma (finished)

Now,

$$\varphi_n = 0,$$

Thus, for $H=\mathbb{K}^{n-1}\times\{0\}$,

$$\forall N \in W, \quad \operatorname{Im} N \subset H \Rightarrow \operatorname{Im} \varphi(N) \subset H.$$

In this, *H may be replaced by any linear hyperplane of $\mathbb{K}^n$.*
The conclusion follows.

Direction for further research: Smaller dimensions

With small finite fields, no expectation for general results like the above one.
$\rightarrow$ We come back to the assumption $\#\mathbb{K} > r$ of Flanders, Atkinson and Lloyd.

Theorem (de Seguins Pazzis, 2013)

Let V maximal subspace of $\mathrm{M}_n(\mathbb{K})$ with $\overline{\mathrm{rk}}(V) = r$, $\dim V \geq nr - 2r + 4$ and $\#\mathbb{K} > r$. Then, V is equivalent to one of the spaces $\mathcal{R}(r, 0)$, $\mathcal{R}(r-1, 1)$, $\mathcal{R}(r-2, 2)$, $\mathcal{R}(2, r-2)$, $\mathcal{R}(1, r-1)$ or $\mathcal{R}(0, r)$.

Naive conjecture

Conjecture

Let V maximal linear subspace of $\mathrm{M}_n(\mathbb{K})$ with upper-rank r and $\dim V \geq nr - \lfloor \frac{r^2}{4} \rfloor$. Then, $V \sim \mathcal{R}(s,t)$ for some (s,t) with $s+t=r$.

Unfortunately ...

The conjecture fails!

A counterexample: the space of all matrices

$$\begin{bmatrix} A & [?]_{3\times(n-3)} \\ [0] & [?]_{(n-3)\times(n-3)} \end{bmatrix}$$

with $A \in \mathrm{A}_3(\mathbb{K})$.

- Upper-rank $n-1$;
- Dimension $n^2 - 3(n-3) - 6 = n(n-1) - 2(n-1) + 1$.
- Maximal.

Primitive matrix spaces

Definition

Let $V \subset \mathrm{M}_{n,p}(\mathbb{K})$ subspace with upper-rank r. We say that V is **non-primitive** if $V \sim W$, where:

- either $\forall M \in W,\; M = \begin{bmatrix} [?]_{n\times(p-1)} & [0]_{n\times 1} \end{bmatrix}$;
- or $\forall M \in W,\; M = \begin{bmatrix} [?]_{(n-1)\times p} \\ [0]_{1\times p} \end{bmatrix}$;
- or $\forall M \in W,\; M = \begin{bmatrix} H(M) & [?]_{n\times 1} \end{bmatrix}$ and $\overline{\mathrm{rk}}\, H(W) = r-1$;
- or $\forall M \in W,\; M = \begin{bmatrix} H(M) \\ [?]_{1\times p} \end{bmatrix}$ and $\overline{\mathrm{rk}}\, H(W) = r-1$;

Examples: $\mathcal{R}(s,t)$ non-primitive if $s+t \leq p$.
The space $\mathrm{A}_n(\mathbb{K})$ is primitive if n odd!

Conjecture

If n odd, then every maximal primitive subspace of $\mathrm{M}_n(\mathbb{K})$ *with upper-rank* $n-1$ *has dimension* $\leq \dfrac{n(n-1)}{2}$.

No valid strategy for this yet!