Range-compatible homomorphisms on matrix spaces

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Notation

- $\mathbb{K}$ arbitrary field;
- $M_{n,p}(\mathbb{K})$ the vector space of $n \times p$ matrices, entries in $\mathbb{K}$;
- $\mathcal{L}(U, V)$ space of all linear maps from U to V (U and V finite-dimensional vector spaces over $\mathbb{K}$);

Let $\mathcal{S}$ linear subspace of $\mathcal{L}(U,V)$. A map

$$F:\mathcal{S}\rightarrow V$$

is **range-compatible** when

$$\forall s\in\mathcal{S},\; F(s)\in\operatorname{Im} s.$$

It is **local** when there exists $x\in U$ s.t.

$$\forall s\in\mathcal{S},\; F(s)=s(x).$$

Every local map is range-compatible! The converse **fails** in general, even for linear maps.
Example:

$$\mathcal{S} := \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid (a,b) \in \mathbb{K}^2 \right\}$$

and

$$F : \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mapsto \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

A known (?) theorem:

Theorem

Every RC linear map on $\mathcal{L}(U,V)$ is local.

The problem:

Does this still hold for large subspaces of $\mathcal{L}(U,V)$?
How large?
What about RC homomorphisms?

In the study of large spaces of bounded rank matrices

Let n,p,r positive integers s.t. $n \geq r$. Let

$$\mathcal{W} \subset \mathrm{M}_{n,r}(\mathbb{K})$$

with $\operatorname{codim} \mathcal{W} \leq n-2$, and $F : \mathcal{W} \rightarrow \mathrm{M}_{n,p}(\mathbb{K})$ lin. map. Assume that every matrix in

$$\mathcal{V} := \left\{ \begin{bmatrix} N & F(N) \end{bmatrix} \mid N \in \mathcal{W} \right\} \subset \mathrm{M}_{n,p+r}(\mathbb{K})$$

has rank $\leq r$. Then,

$$\forall N \in \mathcal{W},\ \operatorname{Im} F(N) \subset \operatorname{Im} N.$$

(this uses Flanders's theorem for affine spaces).

Now, write

$$F(N)=\begin{bmatrix}C_1(N) & \cdots & C_p(N)\end{bmatrix}.$$

Each C_i map is RC!
If one can prove that C_i is local, then using column operations one finds $\mathcal{V}$ equivalent to a subspace of matrices of the form

$$\begin{bmatrix}[?]_{n\times r} & [0]_{n\times p}\end{bmatrix}.$$

Invertibility preservers

Theorem (de Seguins Pazzis, 2012)

Let $\mathcal{S}$ be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with $\operatorname{codim} \mathcal{S} \leq n-2$. Let

$$u : \mathcal{S} \to \mathrm{M}_n(\mathbb{K})$$

an injective linear map s.t.

$$\forall M \in \mathcal{S},\ u(M) \in \mathrm{GL}_n(\mathbb{K}) \Leftrightarrow M \in \mathrm{GL}_n(\mathbb{K}).$$

Then, there exists $(P,Q) \in \mathrm{GL}_n(\mathbb{K})^2$ s.t.

$$u : M \mapsto PMQ \quad \text{or} \quad u : M \mapsto PM^TQ.$$

Sketch of proof if $n \geq 3$

- For $d \in \mathbb{P}(\mathbb{K}^n)$, one sets $E_d := \{M \in \mathrm{M}_n(\mathbb{K}) : \; d \subset \operatorname{Ker} M\}$ and $E^d := (E_d)^T$.
- One proves that, for all $d \in \mathbb{P}(\mathbb{K}^n)$, either $u^{-1}(E_d) = E_{d'} \cap \mathcal{S}$ or $u^{-1}(E_d) = E^{d'} \cap \mathcal{S}$ for some $d' \in \mathbb{P}(\mathbb{K}^n)$, and ditto for the E^d spaces. This uses the Atkinson-Lloyd theorem.
- Composing u with $M \mapsto M^T$ if necessary, one reduces the situation to the one where there are bijections

$$\varphi : \mathbb{P}(\mathbb{K}^n) \to \mathbb{P}(\mathbb{K}^n) \quad \text{and} \quad \psi : \mathbb{P}(\mathbb{K}^n) \to \mathbb{P}(\mathbb{K}^n)$$

such that

$$\forall d \in \mathbb{P}(\mathbb{K}^n), \; u^{-1}(E_d) = E_{\varphi(d)} \cap \mathcal{S} \quad \text{and} \quad u^{-1}(E^d) = E^{\psi(d)} \cap \mathcal{S}.$$

Using the fundamental theorem of projective geometry, one shows that ψ is a homography.
WLOG, $\psi = \mathrm{id}$. Then,

$$\forall M \in \mathcal{S},\ \operatorname{Im} u(M) \subset \operatorname{Im} M.$$

We split

$$u(M) = \begin{bmatrix} F_1(M) & \cdots & F_n(M) \end{bmatrix}$$

Then, $F_1, \dots, F_n$ are RC!
If we know that the F_i's are local, then,

$$u : M \mapsto MQ$$

for some $Q \in \mathrm{M}_n(\mathbb{K})$. Using $\operatorname{codim} \mathcal{S} \leq n-2$, one finds that Q is invertible, QED.

Connection with algebraic reflexivity

Let $\mathcal{S}$ lin. subspace of $\mathcal{L}(U,V)$. Its **reflexive closure** is

$$\mathcal{R}(\mathcal{S}) := \{g \in \mathcal{L}(U,V) : \ \forall x \in U, \ \exists f \in \mathcal{S} : g(x) = f(x)\}.$$

Problem: Find sufficient conditions on $\mathcal{S}$ so that $\mathcal{R}(\mathcal{S}) = \mathcal{S}$ (that is, $\mathcal{S}$ is **algebraically reflexive**).

For $x \in U$, set

$$\hat{x} : s \in \mathcal{S} \mapsto s(x) \in V.$$

Then,

$$\widehat{\mathcal{S}} := \{\widehat{x} \mid x \in U\} \subset \mathcal{L}(\mathcal{S},V)$$

dual operator space.

Consider a linear map

$$F : \widehat{\mathcal{S}} \to V.$$

Then,

$$\check{F} : x \in U \mapsto F(\hat{x}) \in V$$

is linear. One proves that

$$F \text{ local } \Leftrightarrow \check{F} \in \mathcal{S}$$

and

$$F \text{ RC } \Leftrightarrow \check{F} \in \mathcal{R}(\mathcal{S}).$$

Hence, every RC linear map on $\widehat{\mathcal{S}}$ is local iff $\mathcal{R}(\mathcal{S}) = \mathcal{S}$.

More generally, one obtains

$$\mathcal{R}(\mathcal{S})/\mathcal{S} \simeq \mathcal{L}_{\text{rc}}(\widehat{\mathcal{S}}, V)/\mathcal{L}_{\text{loc}}(\widehat{\mathcal{S}}, V).$$

and

$$\mathcal{L}_{\text{rc}}(\mathcal{S}, V)/\mathcal{L}_{\text{loc}}(\mathcal{S}, V) \simeq \mathcal{R}(\widehat{\mathcal{S}})/\widehat{\mathcal{S}}.$$

Main theorems on RC homomorphisms

The first step:

Theorem (de Seguins Pazzis, 2010)

Let $\mathcal{S}$ lin. subspace of $\mathcal{L}(U,V)$ with $\operatorname{codim}\mathcal{S} \leq \dim V - 2$. Then, every RC linear map on $\mathcal{S}$ is local.

This is Lemma 8 from:

C. de Seguins Pazzis, The classification of large spaces of matrices with bounded rank, in press at Israel Journal of Mathematics, arXiv: http://arxiv.org/abs/1004.0298

Generalization:

Theorem (First classification theorem)

Let $\mathcal{S}$ lin. subspace of $\mathcal{L}(U, V)$ with $\operatorname{codim} \mathcal{S} \leq \dim V - 2$.
Then, every RC homomorphism on $\mathcal{S}$ is local.

The bound is optimal for homomorphisms. Let $\varphi : \mathbb{K} \to \mathbb{K}$ non-linear group homomorphism. Then,

$$F : \begin{bmatrix} a & [?]_{1\times(p-1)} \\ [0]_{(n-1)\times 1} & [?]_{(n-1)\times(p-1)} \end{bmatrix} \longmapsto \begin{bmatrix} \varphi(a) \\ [0]_{(n-1)\times 1} \end{bmatrix}$$

range-compatible but non-local.

The optimal bound for linear maps

Theorem (Classification theorem for linear maps)

Let $\mathcal{S}$ lin. subspace of $\mathcal{L}(U, V)$ with codim $\mathcal{S} \leq 2\dim V - 3$ *if* $\#\mathbb{K} > 2$, *and* codim $\mathcal{S} \leq 2\dim V - 4$ *if* $\#\mathbb{K} = 2$.
Then, every RC linear map on $\mathcal{S}$ is local.

See C. de Seguins Pazzis, Range-compatible homomorphisms on matrix spaces, arXiv: http://arxiv.org/abs/1307.3574

A counter-example at the $2\dim V - 2$ threshold

The map

$$\begin{bmatrix} a & b & [?]_{1\times(p-2)} \\ 0 & a & [?]_{1\times(p-2)} \\ [0]_{(n-2)\times 1} & [0]_{(n-2)\times 1} & [?]_{(n-2)\times(p-2)} \end{bmatrix} \longmapsto \begin{bmatrix} b \\ 0 \\ [0]_{(n-2)\times 1} \end{bmatrix}$$

is range-compatible, linear and non-local.

The symmetric case

Theorem

If $\#\mathbb{K} > 2$ then every RC linear map on $S_n(\mathbb{K})$ is local.

Counter-example for $\mathbb{F}_2$. The diagonal map

$$\Delta : M = (m_{i,j}) \in S_n(\mathbb{K}) \mapsto \begin{bmatrix} m_{1,1} \\ \vdots \\ m_{n,n} \end{bmatrix} \in \mathbb{K}^n$$

is RC!
Indeed, for all $X \in \mathbb{F}_2^n$,

$$MX = 0 \Rightarrow X^T MX = 0 \Rightarrow \sum_{k=1}^{n} m_{k,k} x_k^2 = 0 \Rightarrow \Delta(M) \perp X.$$

Theorem

Every RC linear map on $S_n(\mathbb{F}_2)$ is either local or the sum of a local map with Δ.

In particular,

$$\begin{bmatrix} a & b & [?]_{1\times(p-2)} \\ b & c & [?]_{1\times(p-2)} \\ [0]_{(n-2)\times 1} & [0]_{(n-2)\times 1} & [?]_{(n-2)\times(p-2)} \end{bmatrix} \longmapsto \begin{bmatrix} a \\ c \\ [0]_{(n-2)\times 1} \end{bmatrix}$$

is range-compatible, linear and non-local.

The splitting technique

Let $\mathcal{A}$, $\mathcal{B}$ respective lin. subspaces of $M_{n,p}(\mathbb{K})$ and $M_{n,q}(\mathbb{K})$. One sets

$$\mathcal{A} \coprod \mathcal{B} := \Big\{ \begin{bmatrix} A & B \end{bmatrix} \mid A \in \mathcal{A},\ B \in \mathcal{B} \Big\}.$$

Every homomorphism (resp. linear map) F from $\mathcal{A} \coprod \mathcal{B}$ to $\mathbb{K}^n$ splits as

$$f \coprod g : \begin{bmatrix} A & B \end{bmatrix} \mapsto f(A) + g(B)$$

where $f : \mathcal{A} \to \mathbb{K}^n$ and $g : \mathcal{B} \to \mathbb{K}^n$ are homomorphisms (resp. linear maps). Moreover:

- $f \coprod g$ is RC iff f and g are RC;
- $f \coprod g$ is local iff f and g are local.

Lemma

Assume $\dim U = 1$. *Let* $\mathcal{S} \subset \mathcal{L}(U, V)$.
Every RC linear map on $\mathcal{S}$ *is local.*
If $\dim \mathcal{S} \neq 1$, *every RC homomorphism on* $\mathcal{S}$ *is local.*

Proof: We can assume $\mathcal{S} \subset \mathbb{K}^n$. Let $F : \mathcal{S} \rightarrow \mathbb{K}^n$ a RC homomorphism. Then, $\forall X \in \mathcal{S},\ F(X) \in \mathbb{K}X$. Then it is known that $F : X \mapsto \lambda X$ for some fixed λ if F is linear or $\dim \mathcal{S} \neq 1$. $\square$

As a corollary, every RC homomorphism on $\mathcal{L}(U,V)$ is local if $\dim V \geq 2$: indeed if $n \geq 2$ and $p \geq 1$ we split

$$\mathrm{M}_{n,p}(\mathbb{K}) = \mathbb{K}^n \coprod \cdots \coprod \mathbb{K}^n$$

and we know that every RC homomorphism on $\mathbb{K}^n$ is local.

The projection technique

Let V_0 lin. subspace of V, and $\pi : V \twoheadrightarrow V/V_0$ the standard projection. Let $F : \mathcal{S} \to V$ a RC homomorphism. Set

$$\mathcal{S} \operatorname{mod} V_0 := \{\pi \circ s \mid s \in \mathcal{S}\},$$

lin. subspace of $\mathcal{L}(U, V/V_0)$. Then, there is a unique RC homomorphism $F \operatorname{mod} V_0$ on $\mathcal{S} \operatorname{mod} V_0$ s.t.

$$\forall s \in \mathcal{S},\ (F \operatorname{mod} V_0)(\pi \circ s) = \pi(F(s)).$$

Hence, we have a commutative square

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad F \quad} & V \\ {\scriptstyle s \mapsto \pi \circ s} \big\downarrow & & \big\downarrow {\scriptstyle \pi} \\ \mathcal{S} \operatorname{mod} V_0 & \xrightarrow[F \operatorname{mod} V_0]{} & V/V_0. \end{array}$$

Most of the time, one takes $V_0 = \mathbb{K}y$ where y non-zero vector, and one simply writes

$$\mathcal{S} \operatorname{mod} y := \mathcal{S} \operatorname{mod} \mathbb{K}y \quad \text{and} \quad F \operatorname{mod} y := F \operatorname{mod} \mathbb{K}y.$$

Theorem

Let $\mathcal{S}$ lin. subspace of $\mathcal{L}(U,V)$ with $\operatorname{codim} \mathcal{S} \leq \dim V - 2$. Then, every RC homomorphism on $\mathcal{S}$ is local.

The basic idea: **induction on the dimension of V.**
If $\dim V \leq 1$, the result is void. If $\mathcal{S} = \mathcal{L}(U,V)$ and the result is known. In particular we can assume $\dim V > 2$ and $\mathcal{S} \neq \mathcal{L}(U,V)$. A vector $y \in V \setminus \{0\}$ is **good** is

$$\operatorname{codim}(\mathcal{S} \bmod y) \leq \dim(V/\mathbb{K}y) - 2.$$

By the rank theorem, if y is *not* good then $\mathcal{S}$ contains every operator with range $\mathbb{K}y$! As $\mathcal{S} \neq \mathcal{L}(U,V)$:

- The space V has no basis of bad vectors!

Hence, the bad vectors are trapped into some linear hyperplane of V.

This yields linearly independent good vectors y_1 and y_2. By induction each $F \bmod y_i$ is local! This yields x_1, x_2 in U s.t.

$$\forall s \in \mathcal{S},\ F(s) = s(x_i) \mod \mathbb{K}y_i.$$

If $x_1 = x_2$, then $F : s \mapsto s(x_1)$.

Assume that $x_1 \neq x_2$.
Then, $s(x_1 - x_2) \in \text{Vect}(y_1, y_2)$ for all $s \in \mathcal{S}$!
Then, $x_1 - x_2$ extends into a basis of U.
$\mathcal{S}$ represented by $P \coprod \mathrm{M}_{n,p-1}(\mathbb{K})$, where P a 2-dimensional subspace of $\mathbb{K}^n$. Then, by splitting F is local!

We only consider the case when $\#\mathbb{K} > 2$.
Same strategy, induction on $\dim V$**.** Case $\dim V \leq 1$ void.

Case $\dim V = 2$. Then, either $\mathcal{S} = \mathcal{L}(U, V)$, or $\mathcal{S}$ is represented by $D \coprod \mathrm{M}_{2,p-1}(\mathbb{K})$ or by $\mathrm{S}_2(\mathbb{K}) \coprod \mathrm{M}_{2,p-2}(\mathbb{K})$, where $D = \mathbb{K} \times \{0\}$. Then, one uses the splitting lemma.

In the rest, we assume $\dim V \geq 3$.
A non-zero vector $y \in V$ is **good** if

$$\operatorname{codim}(\mathcal{S} \operatorname{mod} y) \leq 2 \dim(V/\mathbb{K}y) - 3.$$

By the rank theorem

$$\operatorname{codim}(\mathcal{S} \operatorname{mod} y) = \operatorname{codim} \mathcal{S} - (\dim U - \dim U')$$

where $U' := \{s \in \mathcal{S} : \operatorname{Im} s \subset \mathbb{K}y\}$.

Consider the orthogonal space

$$\mathcal{S}^\perp := \{t \in \mathcal{L}(V,U) : \forall s \in \mathcal{S},\ \operatorname{tr}(t \circ s) = 0\}.$$

Then,

$$\dim U - \dim U' = \dim(\mathcal{S}^\perp y).$$

Hence,

$$\operatorname{codim}(\mathcal{S} \bmod y) = \operatorname{codim} \mathcal{S} - \dim(\mathcal{S}^\perp y).$$

Consequence:

$$y \text{ bad } \Rightarrow \dim \mathcal{S}^\perp y \leq 1.$$

Claim

The space V has a basis of good vectors, or every RC linear map on $\mathcal{S}$ is local.

Proof of the claimed statement

Assume that there is no basis of V made of good vectors. Then,

$$\operatorname{codim} \mathcal{S} = 2 \dim V - 3.$$

Indeed, if not $\dim \mathcal{S}^{\perp} y = 0$ for every bad vector y, whence $\mathcal{S}^{\perp} = \{0\}$ and $\operatorname{codim} \mathcal{S} \leq 2 \dim V - 5$!

Next, there is a linear hyperplane H of V that contains all the good vectors. Hence, $\dim \mathcal{S}^{\perp} y \leq 1$ for all $y \in V \setminus H$.

Then, every operator in $\widehat{\mathcal{S}^\perp}$ has rank ≤ 1.
Indeed, if the contrary holds we have a quadratic form q on V such that

$$\forall y \in V,\; q(y) \neq 0 \Rightarrow \operatorname{rk} \widehat{y} \geq 2.$$

Then, we choose a non-zero linear form φ on V s.t. $\operatorname{Ker} \varphi = H$, and hence

$$\forall y \in V,\; \varphi(y)\, q(y) = 0.$$

This is absurd since $\# \mathbb{K} > 2$.

Hence

$$\forall t \in \widehat{\mathcal{S}^\perp},\; \operatorname{rk} t \leq 1.$$

Next, one applies the *classification of vector spaces of matrices with rank at most* 1.

Remark 1: No non-zero vector of $\mathcal{S}^\bot$ annihilates all the operators in $\widehat{\mathcal{S}^\bot}$.

Remark 2: $\dim \mathcal{S}^\bot \geq 2$.

$\rightarrow$ there is a line $D \subset U$ that includes the range of every operator in $\widehat{\mathcal{S}^\bot}$;
$\rightarrow$ $\operatorname{Im} t \subset D$ for all $t \in \mathcal{S}^\bot$. Write $D = \mathbb{K}x_1$ and extend x_1 into $(x_1, \dots, x_p)$ basis of U;
$\rightarrow$ $\mathcal{S}$ represented by $W \coprod \mathrm{M}_{n,p-1}(\mathbb{K})$ for some $W \subset \mathbb{K}^n$;

Hence, by splitting every RC linear map on $\mathcal{S}$ is local. QED.

Completing the proof

Assume that $F : \mathcal{S} \rightarrow V$ non-local RC linear map.
We can find linearly independent good vectors y_1, y_2, y_3. By induction, each $F \bmod y_i$ map is local, yielding x_1, x_2, x_3 s.t.

$$\forall i, \forall s,\ F(s) = s(x_i) \mod \mathbb{K} y_i.$$

If $x_i = x_j$ for some distinct i, j then $F : s \mapsto s(x_i)$. Hence, x_1, x_2, x_3 pairwise $\neq$.
WLOG $x_3 = 0$ (replace F with $s \mapsto F(s) - s(x_3)$).
Then, $x_1 \neq 0$, $x_2 \neq 0$ and $x_1 \neq x_2$. Note that

$$\forall s \in \mathcal{S}, \begin{cases} s(x_1) & \in \operatorname{Vect}(y_1, y_3) \\ s(x_2) & \in \operatorname{Vect}(y_2, y_3) \\ s(x_1 - x_2) & \in \operatorname{Vect}(y_1, y_2). \end{cases}$$

If x_1, x_2 collinear then $s(x_1) = s(x_2) = 0$ for all $s \in \mathcal{S}$, and then $F = 0$.
Hence, x_1, x_2 non-collinear.
Consider bases $\mathbf{B} = (x_1, x_2, \cdots)$ and $\mathbf{C} := (y_1, y_2, y_3, \cdots)$. In them, operators in $\mathcal{S}$ represented by matrices of type

$$\begin{bmatrix} a & 0 & [?]_{1\times(p-2)} \\ 0 & c & [?]_{1\times(p-2)} \\ b & b & [?]_{1\times(p-2)} \\ [0]_{(n-3)\times 1} & [0]_{(n-3)\times 1} & [0]_{(n-3)\times(p-2)} \end{bmatrix}.$$

That matrix space has codimension $2n-3$ in $M_{n,p}(\mathbb{K})$!

Then, F corresponds to

$$\begin{bmatrix} a & 0 & [?]_{1\times(p-2)} \\ 0 & c & [?]_{1\times(p-2)} \\ b & b & [?]_{1\times(p-2)} \\ [0]_{(n-3)\times 1} & [0]_{(n-3)\times 1} & [0]_{(n-3)\times(p-2)} \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 0 \\ b \\ [0]_{(n-3)\times 1} \end{bmatrix}.$$

Then,

$$\begin{bmatrix} a & 0 \\ 0 & c \\ b & b \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$$

would be RC! Yet $a = b = c = 1$ shows that this fails. QED.