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Beyond Gerstenhaber: spaces of matrices with spectral conditions

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Introduction

I. The adapted vector method

II. The diagonal-compatibility method

III. Beyond nilpotent subspaces

Introduction

$\mathbb{K}$ denotes an arbitrary (commutative) field (possibly finite), $\mathrm{M}_n(\mathbb{K})$ the vector space of square matrices with n rows, $\mathrm{NT}_n(\mathbb{K})$ the one of strictly upper-triangular matrices. $E_{i,j}$ is the elementary matrix with all entries zero, except an entry 1 at the (i,j)-spot.

Two linear subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathrm{M}_n(\mathbb{K})$ are **similar**, and one writes $\mathcal{V} \simeq \mathcal{W}$, when $\exists P \in \mathrm{GL}_n(\mathbb{K})$ s.t. $\mathcal{W} = P\mathcal{V}P^{-1}$. A linear subspace of $\mathrm{M}_n(\mathbb{K})$ is **nilpotent** when all its elements are. Example: linear subspaces of $\mathrm{NT}_n(\mathbb{K})$ ...

Problem : can one classify the nilpotent subspaces of $\mathrm{M}_n(\mathbb{K})$?

Background: Engel's theorem: every Lie subalgebra of nilpotent matrices of $\mathrm{M}_n(\mathbb{K})$ is triangularizable.

Is every nilpotent subspace of $\mathrm{M}_n(\mathbb{K})$ triangularizable? YES for $n = 2$, NO for $n > 2$. A classical counterexample: the matrices of the form

$$\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -b & a & 0 \end{bmatrix}$$

are all nilpotent; no common eigenvector!

Gerstenhaber's theorem (1958)

If $\mathcal{V}$ is a nilpotent subspace of $\mathrm{M}_n(\mathbb{K})$, then

$$\dim \mathcal{V} \leq \frac{n(n-1)}{2}$$

and equality holds iff $\mathcal{V} \simeq \mathrm{NT}_n(\mathbb{K})$.

M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices (I), *Amer. J. Math.* **80** (1958) 614-622.

Gerstenhaber requires $\#\mathbb{K} \geq n$. For an arbitrary field, proof completed by V.N. Serezhkin:

V.N. Serezhkin, Linear transformations preserving nilpotency (in Russian), *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk* **125** (1985) 46-50.

Simplified proof for:

- the inequality;
- the case of equality if $\#\mathbb{K} > 2$;

in

B. Mathes, M. Omladič, H. Radjavi, Linear spaces of nilpotent matrices, *Linear Algebra Appl.* **149** (1991) 215-225.

An application of Gerstenhaber's theorem: Nilpotency preservers.

E.P. Botta, S. Pierce, W. Watkins, Linear transformations that preserve nilpotent matrices, *Pac. J. Math.* **104** (1983) 39-46.

I. The adapted vector method

Motivation: solve the case of equality. Let $\mathcal{V}$ nilpotent subspace of $\mathrm{M}_n(\mathbb{K})$.
$\mathcal{V}$ seen as a space of endomorphisms of $\mathbb{K}^n$.
One needs a basis $(f_1, \dots, f_n)$ of $\mathbb{K}^n$ in which every element of $\mathcal{V}$ is upper-triangular.
Basic idea: first, find an appropriate f_n! There should be no rank 1 matrix in $\mathcal{V}$ with column space $\mathbb{K} f_n$!

Definition 1. A (non-zero) vector $x \in \mathbb{K}^n$ is **$\mathcal{V}$-adapted** when no rank 1 matrix of $\mathcal{V}$ has column space $\mathbb{K}x$.

A. Existence of adapted vectors

Lemma 1. *Every nilpotent subspace has an adapted vector.*

Even better:

Lemma 2. *Let $\mathcal{V}$ a nilpotent subspace of $\mathrm{M}_n(\mathbb{K})$. Then, one of the vectors of the standard basis $(e_1, \dots, e_n)$ is $\mathcal{V}$-adapted.*

Proof by induction on n. Case $n = 1$ obvious. Assume $n \geq 2$ and no vector of the standard basis is $\mathcal{V}$-adapted. Define $\mathcal{U}$ as the subspace of $\mathcal{V}$ consisting of matrices with last row zero, and write every $M \in \mathcal{U}$ as

$$M = \begin{bmatrix} K(M) & C(M) \\ [0]_{1\times(n-1)} & 0 \end{bmatrix}.$$

Thus, $K(\mathcal{U})$ is a nilpotent subspace of $\mathrm{M}_{n-1}(\mathbb{K})$.

By induction, some e_i $(1 \leq i \leq n-1)$ is $K(\mathcal{U})$-adapted. E.g. e_1 is $K(\mathcal{U})$-adapted. But e_1 is not $\mathcal{V}$-adapted! One finds $R_0 \in \mathrm{M}_{1,n}(\mathbb{K}) \setminus \{0\}$ s.t.

$$\begin{bmatrix} R_0 \\ [0]_{(n-1)\times n} \end{bmatrix} \in \mathcal{V}$$

$R_0 = \begin{bmatrix} 0 & \cdots & 0 & ? \end{bmatrix}$ since e_1 is $K(\mathcal{U})$-adapted. Then, $E_{1,n} \in \mathcal{V}$.

More generally, we have some $i \neq n$ such that $E_{i,n} \in \mathcal{V}$.

More generally, for every $k \in [\![1, n]\!]$, one finds $f(k) \in [\![1, n]\!] \setminus \{k\}$ with $E_{f(k),k} \in \mathcal{V}$.

One chooses an f-cycle $(i_1, \dots, i_p)$, i.e. $i_1, \dots, i_p$ distinct in $[\![1, n]\!]$ and $f(i_1) = i_2, \dots, f(i_{p-1}) = i_p$ and $f(i_p) = i_1$.

Then, $\sum_{k=1}^{p} E_{f(i_k),i_k}$ non-nilpotent in $\mathcal{V}$! QED

B. Using adapted vectors to prove the inequality statement

Without loss of generality, e_n is $\mathcal{V}$-adapted. For $M \in \mathcal{V}$, one splits

$$M = \begin{bmatrix} K(M) & C(M) \\ L(M) & a(M) \end{bmatrix}$$

with $K(M)$ an $(n-1) \times (n-1)$ matrix.

Define $\mathcal{W}$ as the set of $M \in \mathcal{V}$ of the form

$$M = \begin{bmatrix} K(M) & [0]_{(n-1)\times 1} \\ L(M) & a(M) \end{bmatrix}.$$

Then, $K(\mathcal{W})$ is a nilpotent subspace of $\mathrm{M}_{n-1}(\mathbb{K})$.

As e_n is $\mathcal{V}$-adapted, rank theorem yields

$$\dim K(\mathcal{W}) = \dim \mathcal{W}.$$

Thus,

$$\dim \mathcal{V} = \dim C(\mathcal{V}) + \dim K(\mathcal{W}).$$

By induction

$$\dim \mathcal{V} \leq (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}.$$

Remark 1. By induction, one can find a *permutation* matrix P s.t. $P\mathcal{V}P^{-1}$ contains no non-zero lower-triangular matrix!

C. Generalizations and applications

Proposition 3. *Let $\mathcal{V}$ be a linear subspace of* $\mathrm{M}_n(\mathbb{K})$ *in which no matrix has a non-zero eigenvalue. Then $\mathcal{V}$ has an adapted vector.*

Corollary 4. *Let $\mathcal{V}$ be a linear subspace of* $\mathrm{M}_n(\mathbb{K})$ *in which no matrix has a non-zero eigenvalue. Then* $\dim \mathcal{V} \leq \frac{n(n-1)}{2}$.

Similar proofs as for nilpotent subspaces.

Corollary 5. *Let $\mathcal{V}$ be an* affine *subspace of invertible matrices of* $\mathrm{M}_n(\mathbb{K})$. *Then* $\dim \mathcal{V} \leq \frac{n(n-1)}{2}$.

Proof. Denote by V the translation vector space of $\mathcal{V}$. One may assume that $I_n \in \mathcal{V}$. Then, $\alpha I_n - M = \alpha(I_n - \alpha^{-1}M)$ non-singular for all $M \in V$ and all $\alpha \in \mathbb{K} \setminus \{0\}$.
No matrix of V has a non-zero eigenvalue. □

C. de Seguins Pazzis, On the matrices of given rank in a large subspace, *Linear Algebra Appl.* **435-1** (2011) 147-151.

Further successes of the adapted vector method:

Proposition 6. *Let $\mathcal{V}$ a linear subspace of $\mathrm{M}_n(\mathbb{K})$ in which every matrix has ≤ 1 eigenvalue in $\mathbb{K}$.*

Then $\mathcal{V}$ has an adapted vector unless $n = 2$ and $\mathrm{char}(\mathbb{K}) = 2$.

Theorem 7. *With the same assumptions, $\dim \mathcal{V} \leq 1 + \frac{n(n-1)}{2}$.*

Counter-example: $\mathfrak{sl}_2(\mathbb{K})$ with $\mathrm{char}(\mathbb{K}) = 2$.

C. de Seguins Pazzis, Spaces of matrices with a sole eigenvalue, *Lin. Multilin. Alg.* **60** (2012) 1165-1190.

If, in the definition of an adapted vector, one only considers matrices with rank 1 and trace zero:

Proposition 8. *Let $\mathcal{V}$ a linear subspace of* $\mathrm{M}_n(\mathbb{K})$ *in which every matrix has ≤ 2 eigenvalues in $\mathbb{K}$.*
If $n > 2$ and $\operatorname{char}(\mathbb{K}) \neq 2$, then $\mathcal{V}$ has an adapted vector.

Theorem 9. *With the same assumptions,* $\dim \mathcal{V} \leq 2 + \frac{n(n-1)}{2}$.

C. de Seguins Pazzis, Spaces of matrices with few eigenvalues, *arXiv preprint.*

Remark 2. For the "at most 3 eigenvalues" hypothesis, the existence of an adapted vector may fail (as for "at most 2 eigenvalues" when $\operatorname{char}(\mathbb{K}) = 2$).

II. The diagonal-compatibility method

A. Setup

Aim: now that we have found an adapted vector, continue the analysis of the case of equality.

Hypotheses: $\mathcal{V}$ a nilpotent subspace of dimension $\frac{n(n-1)}{2}$ with e_n as adapted vector. Same notation as in the proof of inequality. Then,

$$\dim C(\mathcal{V}) = n-1 \quad \text{and} \quad \dim K(\mathcal{W}) = \frac{(n-1)(n-2)}{2}.$$

By induction $K(\mathcal{W}) \simeq \mathrm{NT}_{n-1}(\mathbb{K})$. Changing the first $n-1$ basis vectors, one may assume

$$K(\mathcal{W}) = \mathrm{NT}_{n-1}(\mathbb{K}).$$

Then, one proves:

Lemma 10. *e_1 is $\mathcal{V}^T$-adapted.*

Proof. Let $C \in \mathrm{M}_{n,1}(\mathbb{K})$ column matrix with $\begin{bmatrix} C & [0]_{n\times(n-1)} \end{bmatrix} \in \mathcal{V}$. As $K(\mathcal{W}) = \mathrm{NT}_{n-1}(\mathbb{K})$,

$$C = \begin{bmatrix} [0]_{(n-1)\times 1} \\ \alpha \end{bmatrix}.$$

As e_n is $\mathcal{V}$-adapted, $\alpha = 0$. □

Then, one use a new splitting for $\mathcal{V}$:

$$M = \begin{bmatrix} a'(M) & L'(M) \\ C'(M) & K'(M) \end{bmatrix}$$

with $K'(M)$ an $(n-1)\times(n-1)$-matrix. Define $\mathcal{W}'$ as the set of all $M \in \mathcal{V}$ with $L'(M) = 0$. Then:

- $K'(\mathcal{W}')$ nilpotent subspace of $\mathrm{M}_{n-1}(\mathbb{K})$;
- As e_1 is $\mathcal{V}^T$-adapted, $\dim K'(\mathcal{W}) = \frac{(n-1)(n-2)}{2}$.

By induction,

$$K'(\mathcal{W}') \simeq \mathrm{NT}_{n-1}(\mathbb{K}).$$

B. Diagonal compatibility

Lemma 11. *The last vector of the standard basis is $K'(\mathcal{W}')$-adapted.*

Proof. Let $L \in \mathrm{M}_{1,n-1}(\mathbb{K})$ s.t. $K'(\mathcal{W}')$ contains $\begin{bmatrix} [0]_{(n-2)\times(n-1)} \\ L \end{bmatrix}$, i.e. for some $C \in \mathrm{M}_{n,1}(\mathbb{K})$,

$$\begin{bmatrix} C & [0]_{(n-1)\times(n-1)} \\ ? & L \end{bmatrix} \in \mathcal{V}$$

Then $C = 0$ since $K(\mathcal{W}) = \mathrm{NT}_{n-1}(\mathbb{K})$. Then, $L = 0$ since e_n is $\mathcal{V}$-adapted. □

Lemma 12 (Diagonal-compatibility)**.** *There is a matrix* $Q = \begin{bmatrix} I_{n-2} & [0]_{(n-2)\times 1} \\ [?]_{1\times(n-2)} & 1 \end{bmatrix}$ *s.t.* $K'(\mathcal{W}') = Q\,\mathrm{NT}_{n-1}(\mathbb{K})Q^{-1}$.

Proof:

Main ideas:

Point 1:

No generality is lost in assuming that $Qe_{n-1} = e_{n-1}$; this is based on the fact that e_{n-1} is $K'(\mathcal{W}')$-adapted.

Point 2:

For every $U \in \mathrm{NT}_{n-2}(\mathbb{K})$,

$$\begin{bmatrix} U & [0]_{(n-2)\times 1} \\ [?]_{1\times(n-2)} & 0 \end{bmatrix} \in K'(\mathcal{W}').$$

Follows from $K(\mathcal{W}) = \mathrm{NT}_{n-1}(\mathbb{K})$.

C. Special matrices, and conclusion

For every $L \in \mathrm{M}_{1,n-2}(\mathbb{K})$, $\mathcal{V}$ contains

$$A_L = \begin{bmatrix} 0 & L & 0 \\ [0]_{(n-2)\times 1} & [0]_{(n-2)\times(n-2)} & [0]_{(n-2)\times 1} \\ f(L) & \varphi(L) & 0 \end{bmatrix}$$

(this uses $K(\mathcal{W}) = \mathrm{NT}_{n-1}(\mathbb{K})$).

For every $C \in \mathrm{M}_{n-2,1}(\mathbb{K})$, $\mathcal{V}$ contains

$$B_C = \begin{bmatrix} 0 & [0]_{1\times(n-2)} & 0 \\ \psi(C) & [0]_{(n-2)\times(n-2)} & C \\ g(C) & [0]_{1\times(n-2)} & 0 \end{bmatrix}$$

(this uses $K'(\mathcal{W}') = \mathrm{NT}_{n-1}(\mathbb{K})$).

For every $U \in \mathrm{NT}_{n-2}(\mathbb{K})$, $\mathcal{V}$ contains

$$E_U = \begin{bmatrix} 0 & [0]_{1\times(n-2)} & 0 \\ [0]_{(n-2)\times 1} & U & [0]_{(n-2)\times 1} \\ h(U) & [0]_{1\times(n-2)} & 0 \end{bmatrix}.$$

Finally, $\mathcal{V}$ contains a matrix of the form

$$J = \begin{bmatrix} ? & [0]_{1\times(n-2)} & 1 \\ [?]_{(n-2)\times 1} & [?]_{(n-2)\times(n-2)} & [0]_{(n-2)\times 1} \\ ? & [?]_{1\times(n-2)} & ? \end{bmatrix}.$$

Then, one successively proves:

- There is a scalar λ such that $\varphi(L) = \lambda L$ and $\psi(C) = -\lambda C$ for all L and C.
- One replaces $\mathcal{V}$ with $P^{-1}\mathcal{V}P$ where
$$P = \begin{bmatrix} 1 & [0]_{1\times(n-2)} & 0 \\ [0]_{(n-2)\times 1} & I_{n-2} & [0]_{(n-2)\times 1} \\ \lambda & [0]_{1\times(n-2)} & 1 \end{bmatrix},$$
and thus one can assume $\lambda = 0$.
- Then, one proves that $f = 0$, $g = 0$ and $h = 0$.
- One shows that $\mathcal{V}$ contains $E_{1,n}$ (this uses J).

One concludes that $\mathcal{V}$ contains $\mathrm{NT}_n(\mathbb{K})$, which completes the proof.

Details in:
C. de Seguins Pazzis, On Gerstenhaber's theorem for spaces of nilpotent matrices over a skew field, *Linear Algebra Appl.* **438-11** (2013) 4426-4438.

III. Beyond nilpotent subspaces

The adapted vector method and the diagonal-compatibility methods yield theorems of the same flavor as Gerstenhaber's.

A. Large spaces of matrices with no non-zero eigenvalue

A matrix $P \in \mathrm{GL}_n(\mathbb{K})$ is **non-isotropic** when $X^T P X \neq 0$ for all non-zero $X \in \mathbb{K}^n$. Two matrices P and Q are **sim-congruent** when there is a non-singular R and a non-zero scalar λ s.t. $P = \lambda R Q R^T$. Two quadratic forms φ and ψ are **similar** when there is a non-zero scalar λ such that ψ is equivalent to $\lambda\,\varphi$.

$\mathrm{A}_n(\mathbb{K})$ is the space of all $n \times n$ alternating matrices.

Theorem 13 (de Seguins Pazzis (2010)). *Let $\mathcal{V}$ be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with dimension $\frac{n(n-1)}{2}$, in which no matrix has a non-zero eigenvalue. Assume that $\#\mathbb{K} > 2$. Then, there are non-isotropic matrices $P_1, \dots, P_p$ respectively in $\mathrm{GL}_{n_1}(\mathbb{K}), \dots, \mathrm{GL}_{n_p}(\mathbb{K})$, s.t. $\mathcal{V}$ is similar to the space of all matrices of the form*

$$\begin{bmatrix} P_1 A_1 & & [?] \\ & \ddots & \\ [0] & & P_p A_p \end{bmatrix}$$

with $A_1 \in \mathrm{A}_{n_1}(\mathbb{K}), \dots, A_p \in \mathrm{A}_{n_p}(\mathbb{K})$.

$P_1, \dots, P_p$ are uniquely determined by $\mathcal{V}$ up to sim-congruence.

Thus, one is reduced to the classification of non-isotropic bilinear forms up to similarity.

C. de Seguins Pazzis, Large affine spaces of non-singular matrices, *Trans. Amer. Math. Soc.* **365** (2013) 2569-2596.

Theorem 14 (de Seguins Pazzis (2010)). *Let $\mathcal{V}$ be an affine subspace of $\mathrm{M}_n(\mathbb{K})$ with dimension $\frac{n(n-1)}{2}$, in which all the matrices are non-singular. Assume $\#\mathbb{K} > 2$.*

Then, there are non-isotropic matrices $P_1, \dots, P_p$ respectively in $\mathrm{GL}_{n_1}(\mathbb{K}), \dots, \mathrm{GL}_{n_p}(\mathbb{K})$, such that $\mathcal{V}$ is equivalent to the space of all matrices of the form

$$I_n + \begin{bmatrix} P_1 A_1 & & [?] \\ & \ddots & \\ [0] & & P_p A_p \end{bmatrix}$$

with $A_1 \in \mathrm{A}_{n_1}(\mathbb{K}), \dots, A_p \in \mathrm{A}_{n_p}(\mathbb{K})$.

The quadratic forms $X \mapsto X^T P_1 X$, …, $X \mapsto X^T P_p X$ are uniquely determined by $\mathcal{V}$ up to similarity.

For a generalization to affine spaces of matrices with a lower bound on the rank:

C. de Seguins Pazzis, Large affine spaces of matrices with rank bounded below, *Linear Algebra Appl.* **437-2** (2012) 499-518.

B. Large spaces of matrices with at most one eigenvalue

Theorem 15. *Let $\mathcal{V}$ be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ in which every matrix has 1 eigenvalue in $\overline{\mathbb{K}}$, and $\dim \mathcal{V} = 1 + \frac{n(n-1)}{2}\cdot$ Then, $\mathcal{V} \simeq \mathbb{K} I_n \oplus \mathrm{NT}_n(\mathbb{K})$ except in the following situations:*

- $\mathrm{char}(\mathbb{K}) = 2$ *and* $n \in \{2, 4\}$;
- $\mathrm{char}(\mathbb{K}) = 3$ *and* $n = 3$.

In the exceptional cases stated above, all the solutions are known.

C. de Seguins Pazzis, Spaces of matrices with a sole eigenvalue, op.cit.

Remark 3. If $\mathrm{char}(\mathbb{K}) \not| \, n$, the result is a trivial consequence of Gerstenhaber's theorem.

Conjecture 1. *Let $\mathcal{V}$ be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ in which every matrix has ≤ 1 eigenvalue in $\mathbb{K}$ and $\dim \mathcal{V} = 1 + \frac{n(n-1)}{2}\cdot$ Assume that $n \geq 5$ and $\#\mathbb{K} > 2$. Then, $\mathcal{V} = \mathbb{K} I_n \oplus \mathcal{H}$, where no matrix of $\mathcal{H}$ has a non-zero eigenvalue in $\mathbb{K}$.*

C. Large spaces of matrices with at most two eigenvalues

For $\mathrm{char}(\mathbb{K}) \neq 2$ and $n \geq 3$, the full classification of spaces of matrices of $\mathrm{M}_n(\mathbb{K})$ with ≤ 2 eigenvalues in $\overline{\mathbb{K}}$ and with the maximal dimension $2 + \frac{n(n-1)}{2}$:

C. de Seguins Pazzis, Spaces of matrices with few eigenvalues, *op.cit.*

Conclusion

$\rightarrow$ Uses only low-level tools from linear algebra.

$\rightarrow$ Well-suited for proving very general results for properties on matrix spaces with spectral conditions.

$\rightarrow$ Main drawback: big machinery, *very* long proofs!