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# Large spaces of matrices with bounded rank

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# Introduction

$\mathbb{K}$ denotes an arbitrary (commutative) field (possibly finite), $n$ and $p$ two positive integers with $n \geq p$, $\mathrm{M}_{n,p}(\mathbb{K})$ the vector space of matrices with $n$ rows, $p$ columns and entries in $\mathbb{K}$, and $\mathrm{M}_n(\mathbb{K}) := \mathrm{M}_{n,n}(\mathbb{K})$.

Two linear subspaces $V$ and $W$ of $\mathrm{M}_{n,p}(\mathbb{K})$ are **equivalent** when $\exists (P,Q) \in \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_p(\mathbb{K})$ such that $W = PVQ$.

The **upper rank** of a subset $V$ of $\mathrm{M}_{n,p}(\mathbb{K})$ is

$$\overline{\mathrm{rk}}(V) := \max\big\{\mathrm{rk}\, M \mid M \in V\big\}.$$

**Problem :** given an integer $r > 0$ with $r < n$ and $r < p$, classify the linear subspaces of $\mathrm{M}_{n,p}(\mathbb{K})$ with upper rank $r$.

For small values of $r$, the solution is known (Schur: $r = 1$; Atkinson: $2 \leq r \leq 3$). For general values, no fully encompassing answer is known.

More realistic goals:

(i) Find the maximal dimension for a subspace with upper rank $r$.

(ii) Classify those subspaces with a dimension "close to" the maximal one.

# I. Historical results

## A. Maximal dimension

**Theorem 1** (Flanders, 1962). *Let* $r \in [\![1, p-1]\!]$ *and* $V$ *be a linear subspace of* $\mathrm{M}_n(\mathbb{K})$ *with* $\overline{\mathrm{rk}}(V) = r$*. Then:*

*(a)* $\dim V \leq rn$ *;*

*(b) If* $\dim V = rn$ *and* $n > p$*, then* $V$ *is equivalent to*

$$\left\{ \begin{bmatrix} C_1 & \cdots & C_r & [0]_{n\times(p-r)} \end{bmatrix} \mid C_1, \dots, C_r \in \mathrm{M}_{n,1}(\mathbb{K}) \right\}$$

*(c) If* $\dim V = rn$ *and* $n = p$*, then* $V$ *or* $V^T$ *is equivalent to*

$$\left\{ \begin{bmatrix} C_1 & \cdots & C_r & [0]_{n\times(n-r)} \end{bmatrix} \mid C_1, \dots, C_r \in \mathrm{M}_{n,1}(\mathbb{K}) \right\}$$

The case $\#\mathbb{K} > r$:

H. Flanders, On spaces of linear transformations with bounded rank, *J. Lond. Math. Soc.* **37** (1962) 10-16.

The case $n = p$ and $r = n - 1$ for an arbitrary field (extended to affine subspaces):

J. Dieudonné, Sur une généralisation du groupe orthogonal à quatre variables, *Arch. Math.* **1** (1949) 282-287.

The more general case:

R. Meshulam, On the maximal rank in a subspace of matrices, *Quart. J. Math. Oxford (2)* **36** (1985) 225-229.

The more general case extended to affine subspaces:

C. de Seguins Pazzis, The affine preservers of non-singular matrices, *Arch. Math.* **95** (2010) 333-342.

## **B.** Below the maximal dimension

The naive conjecture: a subspace $V$ with upper rank $r$ may be extended to one with the maximal dimension $nr$. Equivalently, if $n > p$, then the matrices of $V$ vanish on some common $(p-r)$-dimensional subspace of $\mathbb{K}^p$ (if $n = p$, then this should hold for $V$ or for $V^T$).

True for $r = 1$ (Schur)! False in general!

**Notation 1.** Given $(s,t) \in [\![0,n]\!] \times [\![0,p]\!]$, one defines $\mathcal{R}(s,t)$ as the set of all $n \times p$ matrices of the form

$$\begin{bmatrix} [?]_{s\times t} & [?]_{s\times (p-t)} \\ [?]_{(n-s)\times t} & [0]_{(n-s)\times (p-t)} \end{bmatrix}.$$

Note that $\overline{\mathrm{rk}}\big(\mathcal{R}(s,t)\big) \leq s+t$, and equality holds whenever $s+t \leq p$.

In particular, if $r \geq 1$, then $\mathcal{R}(1,r-1)$ is a counter-example to the above conjecture, with dimension
$n(r-1)+(p-r+1) = nr-(n-p+r-1)$ (in particular: $nr-(r-1)$ if $n=p$).

Atkinson and LLoyd ask: can we classify spaces $V$ with upper rank $r$ and

$$nr-(n-p+r-1) \leq \dim V \leq nr \quad ?$$

**Theorem 2** (Atkinson and Lloyd, 1980)**.**
*Let $r \in [\![1,n-1]\!]$ et $V$ be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with $\overline{\mathrm{rk}}(V)=r$.*
*Assume that $\#\mathbb{K} > r$ and $\dim V \geq nr-r+1$.*
*Then:*

*(a) Either $V$ is the subspace of a linear subspace $W$ of $\mathrm{M}_n(\mathbb{K})$ with $\overline{\mathrm{rk}}(W)=r$ and $\dim W = nr$;*

*(b) Or $V$ is equivalent to $\mathcal{R}(1,r-1)$ or $\mathcal{R}(r-1,1)$.*

M.D. Atkinson, S. Lloyd, Large spaces of matrices of bounded rank, *Quart. J. Math. Oxford (2),* **31** (1980) 253-262.

The case $n > p$:

L.B. Beasley, Null spaces of spaces of matrices of bounded rank, in *Current Trends in Matrix Theory* 45-50, Elsevier, 1987.

What about "small" finite fields?

An extension of Atkinson-Lloyd to an arbitrary field was deemed problematic: counter-example for $\mathbb{F}_2$: the space of all upper-triangular matrices with zero trace:

$$\left\{ \begin{bmatrix} a & c & d \\ 0 & b & e \\ 0 & 0 & a+b \end{bmatrix} \mid a, b, c, d, e \in \mathbb{F}_2 \right\}$$

fits into neither of the above categories.

And then no significant new result until:

**Theorem 3** (de Seguins Pazzis, 2010). *Atkinson and Lloyd's theorem holds for an arbitrary field, except when* $n = 3$*,* $r = 2$*,* $\#\mathbb{K} = 2$ *and* $\dim V = 5$*, in which case the above counter-example is the only exception up to equivalence.*

C. de Seguins Pazzis, The classification of large spaces of matrices with bounded rank, arXiv: http://arxiv.org/abs/1004.0298

Beasley's results are also successfully extended to an arbitrary field (no exception!).

# II. Flanders's method

To simplify things, we take $n = p$.

## A. The original method.

Let $V$ a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with $\overline{\mathrm{rk}}(V) = r \in [\![1, n-1]\!]$. One may assume that $V$ contains

$$J_r := \begin{bmatrix} I_r & [0]_{r \times (n-r)} \\ [0]_{(n-r) \times r} & [0]_{(n-r) \times (n-r)} \end{bmatrix}.$$

In that case, we split every matrix of $V$ with the same block sizes:

$$M = \begin{bmatrix} A(M) & C(M) \\ B(M) & D(M) \end{bmatrix}.$$

Assume that $\#\mathbb{K} > r$ and let $M \in V$. Now, given an indeterminate $t$, compute the $(n-r) \times (n-r)$ matrix $P(t)$ of all $(r+1) \times (r+1)$ minors of $J_r + tM$ where one takes all the first $r$ rows and columns:

→ $P(t)$ is a polynomial of $t$ with coefficients in $\mathrm{M}_{n-r}(\mathbb{K})$ and degree $\leq r$;

→ $P(t)$ vanishes everywhere on $\mathbb{K}$, therefore $P(t) = 0$ ;

→ Its coefficient before $t^r$ is $D(M)$; thus

$$D(M) = 0 \quad ;$$

→ Consequently, its coefficient before $t^{r-1}$ is $-B(M)C(M)$; thus

$$B(M)C(M) = 0.$$

From $\forall M \in V,\ B(M)C(M) = 0$, one easily proves that $\dim B(V) + \dim C(V) \leq r(n-r)$ (e.g. $B(V) \times C(V)$ is totally isotropic for the non-degenerate quadratic form $(B, C) \mapsto \mathrm{tr}(BC)$). Therefore:

$$\boxed{\dim V \leq \dim A(V) + \dim B(V) + \dim C(V) \leq nr.}$$

**Analysing the case of equality.**
Assume now $\dim V = nr$. Then (rank theorem) $V$ contains, for every invertible $P \in \mathrm{M}_n(\mathbb{K})$,

$$M_P := \begin{bmatrix} P & [0]_{r\times(n-r)} \\ [0]_{(n-r)\times r} & [0]_{(n-r)\times(n-r)} \end{bmatrix}.$$

With the above method, one replaces $J_r$ with $M_P$ and finds:

$$\forall M \in V,\ B(M)P^{-1}C(M) = 0.$$

Then varying $P$ yields:

$$\forall M \in V,\ B(M) = 0 \quad \text{or} \quad C(M) = 0.$$

One easily concludes: either $B(V) = \{0\}$ or $C(V) = \{0\}$. This yields the Flanders theorem.

## B. Our new approach

Same starting point, assumptions (except on the cardinality of $\mathbb{K}$) and notations. Also, instead of assuming that $V$ contains $J_r$, we simply assume that it contains

$$\begin{bmatrix} Q & [0]_{r\times(n-r)} \\ [0]_{(n-r)\times r} & [0]_{(n-r)\times(n-r)} \end{bmatrix}$$

for some *invertible* $Q \in \mathrm{M}_r(\mathbb{K})$.

We take an arbitrary matrix of $\mathrm{M}_n(\mathbb{K})$

$$M_0 = \begin{bmatrix} P & C(M_0) \\ B(M_0) & D(M_0) \end{bmatrix}.$$

We actually do not compute the matrix of minors of a general sum $M_0 + M$ with $M \in V$ (too complicated!). Rather, we consider a tower of linear subspaces of $V$:

$$V_3 \subset V_2 \subset V_1 \subset V.$$

where

$$\begin{cases} V_1 & := \operatorname{Ker} A \\ V_2 & := \operatorname{Ker} A \cap \operatorname{Ker} B \\ V_3 & = \operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C. \end{cases}$$

The rank theorem shows:

$$\dim V = \dim A(V) + \dim B(V_1) + \dim C(V_2) + \dim D(V_3).$$

One then computes the matrix of all the $(r+1) \times (r+1)$ minors of $M_0 + M$ which use the $r$ first rows and columns, for $M \in V_1$: this is

$$\det(P) D(M_0 + M) \\ - B(M_0 + M) \operatorname{com}(P)^T C(M_0 + M).$$

This matrix is 0, and so is the one for $M = 0$. Subtracting both equalities yields the **fundamental formula**:

$$B(M) \operatorname{com}(P)^T c(M) = \det(P) D(M) \\ - B(M) \operatorname{com}(P)^T c(M_0) \\ - B(M_0) \operatorname{com}(P)^T c(M)$$

As $P$ may be chosen invertible, this yields:

$$\boxed{D(V_3) = 0.}$$

Taking the polar form of the left-hand side yields, for every $(M, N) \in V_1^2$:

$$B(M)\,\mathrm{com}(P)^T c(N) + B(N)\,\mathrm{com}(P)^T c(M) = 0.$$

In particular, for every $(M, N) \in V_1 \times V_2$:

$$B(M)\,\mathrm{com}(P)^T c(N) = 0.$$

In particular
$\forall (M, N) \in V_1 \times V_2,\ B(M)Q^{-1}C(N) = 0$, and therefore

$$\boxed{\dim B(V_1) + \dim C(V_2) \leq r(n - r).}$$

This yields $\dim V \leq nr$.

For the case $\dim V = nr$, the proof goes on using the above formulas …

# III. Sketch of the proof for the generalized Atkinson-Lloyd theorem.

We continue with the assumptions of the latest section. We now assume $\dim V > nr - r + 1$.

## **A.** Step 1: showing that $B(V_1) = 0$ or $C(V_2) = \{0\}$.

We show that either there is no non-zero matrix in $V$ of the form

$$\begin{bmatrix} [0]_{r\times n} \\ [?]_{(n-r)\times n} \end{bmatrix}$$

or there is no non-zero matrix in $V$ of the form

$$\begin{bmatrix} [0]_{n\times r} & [?]_{n\times(n-r)} \end{bmatrix}.$$

Using the above inequalies, we find

$$\dim A(V) > r^2 - (r-1).$$

On the other hand:

$$\forall P \in A(V),\ \forall (M,N) \in V_1 \times V_2,$$
$$B(M)\,\mathrm{com}(P)^T C(N) = 0.$$

**Lemma 4.** *One has* $B(V_1) = \{0\}$ *or* $C(V_2) = \{0\}$.

*Proof.* Assume the contrary holds. Then $\mathrm{span}\{\mathrm{com}(P)^T \mid P \in A(V)\}$ does not act transitively on $\mathbb{K}^r$. Without loss of generality, one may then assume that

$$\forall P \in A(V),\ \mathrm{com}(P)_{r,r} = 0.$$

This contradicts the Flanders theorem!! □

If $B(V_1) = \{0\}$, then for $W := V^T$, one has $C(W_2) \subset B(V_1)^T = \{0\}$. No generality is then lost by assuming that $C(V_2) = \{0\}$. Then $V_2 = V_3 = \{0\}$.

**B.** Step 2: Finding a common null-space.

We now assume $V_2 = \{0\}$. In that case, there is a linear subspace $W \subset \mathrm{M}_{n,r}(\mathbb{K})$ and a linear map $\varphi : W \to \mathrm{M}_{n,n-r}(\mathbb{K})$ s.t.

$$V = \left\{ \begin{bmatrix} M & \varphi(M) \end{bmatrix} \mid M \in W \right\}.$$

Note that $\operatorname{codim} W < r - 1$.

From there, we prove two lemmas.

**Lemma 5.** *One has*

$$\forall N \in W,\ \operatorname{Im} \varphi(N) \subset \operatorname{Im}(N).$$

**Lemma 6.** *There exists $C \in \mathrm{M}_{r,n-r}(\mathbb{K})$ s.t. $\forall N \in W,\ \varphi(N) = NC$.*

With the last one, take $P := \begin{bmatrix} I_r & -C \\ [0]_{(n-r)\times r} & I_{n-r} \end{bmatrix}$ and check that any matrix of $VP$ has all last $n - r$ columns equal to zero.

**Proof of Lemma 5:** Denote by $U$ the space of matrices of $W$ with 0 as last column. For $M \in U$, write

$$\begin{bmatrix} M & \varphi(M) \end{bmatrix} = \begin{bmatrix} K(M) & [?]_{(n-1)\times(n-r)} \\ [0]_{1\times r} & \varphi_n(K(M)). \end{bmatrix}$$

Note that:

- $\rightarrow$ $\operatorname{codim} K(U) < r-1 < n-2$;
- $\rightarrow$ if $\operatorname{rk} K(M) = r$, then $\varphi_n(K(M)) = 0$.
- $\rightarrow$ $\varphi_n : K(U) \rightarrow \mathrm{M}_{1,n-r}(\mathbb{K})$ is linear.

We deduce that $\varphi_n = 0$ by using the following corollary of our extension of Flander's theorem to affine subspaces:

**Corollary 7.** *Let $V'$ be a linear subspace of $\mathrm{M}_{n,p}(\mathbb{K})$, with $n \geq p \geq r$, and assume that $\dim V' > rn$. Then $V'$ is spanned by its matrices of rang $\geq r$, unless $n = p = 2$, $r = 1$ and $\#\mathbb{K} = 2$.*

*Proof.* If the conclusion does not hold, then take a linear hyperplane $H$ of $V'$ which contains every rank $\geq r$ matrices of $V'$, and then a parallel disjoint affine hyperplane $H'$, which then has dimension $\geq rn$, has upper rank $< r$ and is not a linear subspace of $\mathrm{M}_{n,r}(\mathbb{K})$. $\square$

Back to the proof of Lemma 5: we now have $\varphi_n = 0$. This means that for $H = \mathbb{K}^{n-1} \times \{0\}$, $\forall M \in W,\ \operatorname{Im} M \subset H \Rightarrow \operatorname{Im} \varphi(M) \subset H$. However, *$H$ may be replaced by any linear hyperplane of $\mathbb{K}^n$.* The conclusion follows.

Lemma 6 may be extended as some kind of linear preserver theorem:

**Theorem 8** (Representation lemma)**.** *Let $m, n, p$ be positive integers, $W$ be a linear subspace of $\mathrm{M}_{m,n}(\mathbb{K})$ with $\operatorname{codim} W \leq m-2$, and $\varphi : W \to \mathrm{M}_{m,p}(\mathbb{K})$ be a linear map s.t.*

$$\forall M \in W,\ \operatorname{Im} \varphi(M) \subset \operatorname{Im} M.$$

*Then there exists* $C \in \mathrm{M}_{n,p}(\mathbb{K})$ *s.t.*

$$\varphi : M \mapsto MC.$$

*Remarks* 1.

- The case $W = \mathrm{M}_{m,p}(\mathbb{K})$ is an easy exercise.
- One need only assume $p = 1$.
- The condition $\operatorname{codim} W \leq m - 2$ is not optimal. When $\#\mathbb{K} > 2$ and $n \geq 2$, the best upper bound is $2m - 3$ (work in progress ...).

The proof is a bit too technical for a seminar talk ...

# IV. Application to an invertibility preserver problem.

The following theorem is the first application of the Atkinson-Lloyd theorem.

**Theorem 9** (de Seguins Pazzis, 2012). *Let $V$ be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with codimension $\leq n-2$, and $f : V \hookrightarrow \mathrm{M}_n(\mathbb{K})$ be a linear embedding which is a strong invertibility preserver. Then, unless $n = 3$, $\#\mathbb{K} = 2$ and* $\operatorname{codim} V = 1$*, there exists $(P, Q) \in \mathrm{GL}_n(\mathbb{K})^2$ s.t.*

$$\forall M,\ f(M) = PMQ \quad \text{or} \quad \forall M,\ f(M) = PM^TQ$$

*If $\mathbb{K}$ is infinite or $f(V) = V$, then one need only assume that $f$ is a weak invertibility preserver.*

This is a grand generalization of the well-known Dieudonné theorem (ibid). The method is similar, with the Atkinson-Lloyd theorem as a key starting point. The above representation theorem is also a crucial point.

C. de Seguins Pazzis, The linear preservers of non-singularity in a large space of matrices, *Linear Algebra Appl.* **436-9** (2012), 3507-3530.

*Remark* 2. In the case $n > p$, invertibility is naturally replaced with the assumption "$\operatorname{rk} M = p$" i.e. "$M$ has full rank": similar results hold (currently being written).