# Continuous space-time transformations

Clément de Seguins Pazzis, Peter Šemrl

KU Leuven 2016

# The setting

$V = \mathbb{R}^4$ equipped with the Lorentz quadratic form

$$q : (x, y, z, t) \mapsto t^2 - x^2 - y^2 - z^2.$$

Space-time events $a$ and $b$ are **coherent** iff

$$q(b - a) = 0,$$

they are **adjacent** iff coherent and $a \neq b$.

# Light cones

- Basic light cone:

$$\mathcal{C}(0) := \{m \in V : \; q(m) = 0\}.$$

- Light cone with vertex $a \in V$:

$$\mathcal{C}(a) = \{m \in V : \; q(m-a) = 0\} = a + \mathcal{C}(0).$$

*Critical note:* the maximal coherent sets are *lines* in light cones.

# Light cones

- Basic light cone:

$$\mathcal{C}(0) := \{m \in V : \; q(m) = 0\}.$$

- Light cone with vertex $a \in V$:

$$\mathcal{C}(a) = \{m \in V : \; q(m-a) = 0\} = a + \mathcal{C}(0).$$

*Critical note:* the maximal coherent sets are *lines* in light cones.

# Alexandrov's problem

- **Q:** What are the bijections $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ that preserve coherency in both directions?
- **A:** Only the **standard** maps (or **Poincaré similarities**):

$$m \mapsto \lambda u(m) + a$$

with $a \in \mathbb{R}^4$, $\lambda \in \mathbb{R} \setminus \{0\}$, $u : \mathbb{R}^4 \to \mathbb{R}^4$ a linear *q-isometry*.
- *Note:* this generalizes to an $n$-dimensional **Alexandrov space**, i.e. real quadratic space $(V, q)$, with $q$ regular (i.e. non-degenerate) of signature $(1, n-1)$, with $n \geq 4$.

# Alexandrov's problem

- **Q:** What are the bijections $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ that preserve coherency in both directions?
- **A:** Only the **standard** maps (or **Poincaré similarities**):

$$m \mapsto \lambda u(m) + a$$

with $a \in \mathbb{R}^4$, $\lambda \in \mathbb{R} \setminus \{0\}$, $u : \mathbb{R}^4 \to \mathbb{R}^4$ a linear *q-isometry*.

- *Note:* this generalizes to an $n$-dimensional **Alexandrov space**, i.e. real quadratic space $(V, q)$, with $q$ regular (i.e. non-degenerate) of signature $(1, n-1)$, with $n \geq 4$.

# Alexandrov's problem

- **Q:** What are the bijections $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ that preserve coherency in both directions?
- **A:** Only the **standard** maps (or **Poincaré similarities**):

$$m \mapsto \lambda u(m) + a$$

with $a \in \mathbb{R}^4$, $\lambda \in \mathbb{R} \setminus \{0\}$, $u : \mathbb{R}^4 \to \mathbb{R}^4$ a linear *q-isometry*.

- *Note:* this generalizes to an $n$-dimensional **Alexandrov space**, i.e. real quadratic space $(V, q)$, with $q$ regular (i.e. non-degenerate) of signature $(1, n-1)$, with $n \geq 4$.

# Matrix formulation

- A model of 4-dimensional Alexandrov space:

$$\mathcal{H}_2 = \left\{ \begin{bmatrix} a & \overline{c} \\ c & b \end{bmatrix} \mid (a,b) \in \mathbb{R}^2,\ c \in \mathbb{C} \right\},$$

with

$$q : M \in \mathcal{H}_2 \mapsto \det M.$$

$A$ and $B$ adjacent iff $\mathrm{rk}(A - B) = 1$.

- Standard maps:

$$M \mapsto \epsilon P M P^\star + A, \quad A \in \mathcal{H}_2,\ P \in \mathrm{GL}_2(\mathbb{C}),\ \epsilon \in \{1, -1\}$$

$$M \mapsto \epsilon P M^T P^\star + A, \quad A \in \mathcal{H}_2,\ P \in \mathrm{GL}_2(\mathbb{C}),\ \epsilon \in \{1, -1\}.$$

# Matrix formulation

- A model of 4-dimensional Alexandrov space:

$$\mathcal{H}_2 = \left\{ \begin{bmatrix} a & \overline{c} \\ c & b \end{bmatrix} \mid (a,b) \in \mathbb{R}^2,\ c \in \mathbb{C} \right\},$$

  with

$$q : M \in \mathcal{H}_2 \mapsto \det M.$$

  $A$ and $B$ adjacent iff $\mathrm{rk}(A - B) = 1$.

- Standard maps:

$$M \mapsto \epsilon PMP^\star + A, \quad A \in \mathcal{H}_2,\ P \in \mathrm{GL}_2(\mathbb{C}),\ \epsilon \in \{1, -1\}$$

$$M \mapsto \epsilon PM^T P^\star + A, \quad A \in \mathcal{H}_2,\ P \in \mathrm{GL}_2(\mathbb{C}),\ \epsilon \in \{1, -1\}.$$

# Problem

What if we relax the assumptions?

Drop bijectivity?

Assume the preservation of coherency in one direction only?

# Problem

What if we relax the assumptions?

Drop bijectivity?

Assume the preservation of coherency in one direction only?

# Problem

What if we relax the assumptions?

Drop bijectivity?

Assume the preservation of coherency in one direction only?

# A theorem of Huang and Šemrl

Let $\phi : \mathcal{H}_2 \to \mathcal{H}_2$ preserve adjacency in one direction only. Then:

- either $\phi$ is standard;
- or $\phi$ maps $\mathcal{H}_2$ into a *line* (**degenerate** adjacency preserver).

Example of a degenerate preserver:

$$\phi : A \mapsto \begin{bmatrix} \mathrm{tr}(A) & 0 \\ 0 & 0 \end{bmatrix}.$$

The trace can be replaced with an injective map . . .
Note: generalization to maps $\mathcal{H}_n \to \mathcal{H}_n$.

W.-l. Huang, P. Šemrl,
Adjacency preserving maps on hermitian matrices,
Canad. J. Math. **60** (2008), 1050–1066.

# A theorem of Huang and Šemrl

Let $\phi : \mathcal{H}_2 \to \mathcal{H}_2$ preserve adjacency in one direction only. Then:

- either $\phi$ is standard;
- or $\phi$ maps $\mathcal{H}_2$ into a *line* (**degenerate** adjacency preserver).

Example of a degenerate preserver:

$$\phi : A \mapsto \begin{bmatrix} \mathrm{tr}(A) & 0 \\ 0 & 0 \end{bmatrix}.$$

The trace can be replaced with an injective map ...

Note: generalization to maps $\mathcal{H}_n \to \mathcal{H}_n$.

W.-l. Huang, P. Šemrl,
Adjacency preserving maps on hermitian matrices,
Canad. J. Math. **60** (2008), 1050–1066.

# A theorem of Huang and Šemrl

Let $\phi : \mathcal{H}_2 \to \mathcal{H}_2$ preserve adjacency in one direction only. Then:

- either $\phi$ is standard;
- or $\phi$ maps $\mathcal{H}_2$ into a *line* (**degenerate** adjacency preserver).

Example of a degenerate preserver:

$$\phi : A \mapsto \begin{bmatrix} \mathrm{tr}(A) & 0 \\ 0 & 0 \end{bmatrix}.$$

The trace can be replaced with an injective map ...
Note: generalization to maps $\mathcal{H}_n \to \mathcal{H}_n$.

W.-l. Huang, P. Šemrl,
Adjacency preserving maps on hermitian matrices,
Canad. J. Math. **60** (2008), 1050–1066.

Problem: what if we only assume that *coherency* is preserved (in one direction only)?

Very hard problem . . .

We add *continuity*.

Reformulation: describe the continuous maps $\phi : V \to V$ s.t.

$$\forall (a,b) \in V^2,\ q(a-b)=0 \Rightarrow q\big(\phi(a)-\phi(b)\big)=0.$$

Problem: what if we only assume that *coherency* is preserved (in one direction only)?

Very hard problem . . .

We add *continuity*.

Reformulation: describe the continuous maps $\phi : V \to V$ s.t.

$$\forall (a, b) \in V^2,\ q(a - b) = 0 \Rightarrow q\big(\phi(a) - \phi(b)\big) = 0.$$

Problem: what if we only assume that *coherency* is preserved (in one direction only)?

Very hard problem . . .

We add *continuity*.

Reformulation: describe the continuous maps $\phi : V \to V$ s.t.

$$\forall (a,b) \in V^2,\ q(a-b)=0 \Rightarrow q\big(\phi(a)-\phi(b)\big)=0.$$

Problem: what if we only assume that *coherency* is preserved (in one direction only)?

Very hard problem ...

We add *continuity*.

Reformulation: describe the continuous maps $\phi : V \to V$ s.t.

$$\forall (a,b) \in V^2,\ q(a-b)=0 \Rightarrow q\big(\phi(a)-\phi(b)\big)=0.$$

# Main result

**Theorem (dSP, Šemrl (2015))**

*If $\phi$ is a continuous coherency preserver then:*

- *either $\phi$ is standard;*
- *or $\phi(V) \subset \mathcal{C}(a)$ for some $a \in V$ (degenerate preserver).*

C. de Seguins Pazzis, P. Šemrl,
Continuous space-time transformations,
arXiv: http://arxiv.org/abs/1502.01149

# Surprise: Not all degenerate continuous coherency preservers map into a line!

Choose a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of $V$ s.t.

$$\forall (m,n)\in\mathbb{N}^2,\; m\neq n \Rightarrow \forall (a,b)\in U_n\times U_m,\; q(a-b)\neq 0.$$

Choose:

- $a\in V$;
- For each $n\in\mathbb{N}$, choose $x_n\in\mathcal{C}(0)$ with $\|x_n\|=1$;
- $\alpha : V\rightarrow\mathbb{R}$ continuous with support in $\overline{\bigcup_{n\in\mathbb{N}} U_n}$;

Take

$$\phi : m\mapsto \begin{cases} a+\alpha(m)x_n & \text{if } m\in U_n \\ a & \text{otherwise} \end{cases}$$

Surprise: Not all degenerate continuous coherency preservers map into a line!

Choose a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of $V$ s.t.

$$\forall (m,n)\in\mathbb{N}^2,\ m\neq n \Rightarrow \forall (a,b)\in U_n\times U_m,\ q(a-b)\neq 0.$$

Choose:

- $a\in V$;
- For each $n\in\mathbb{N}$, choose $x_n\in\mathcal{C}(0)$ with $\|x_n\|=1$;
- $\alpha : V\to\mathbb{R}$ continuous with support in $\overline{\bigcup_{n\in\mathbb{N}} U_n}$;

Take

$$\phi : m\mapsto \begin{cases} a+\alpha(m)x_n & \text{if } m\in U_n \\ a & \text{otherwise} \end{cases}$$

# Example of a sequence $(U_n)_{n\in\mathbb{N}}$

With the standard Euclidian norm $\|-\|$ on $\mathbb{R}^4$,

$$U_n := B_o\Big((0_{\mathbb{R}^3}, n), \frac{1}{4}\Big).$$

For $(x,s) \in U_n$ and $(y,t) \in U_p$ with $n \neq p$ (and $(s,t) \in \mathbb{R}^2$, $(x,y) \in (\mathbb{R}^3)^2$),

$$(s-t)^2 > \frac{1}{4} > \|x-y\|^2$$

whence

$$q\big((x,s)-(y,t)\big) > 0.$$

**Theorem (dSP, Šemrl (2015))**

*Any degenerate continuous coherency preserver is of the previous type.*

Let $\phi : V \to V$ be a non-degenerate continuous coherency preserver.

Basic strategy: prove that $\phi$ preserves adjacency;

Basic technique: analyze the action on light cones.

Let $\phi : V \to V$ be a non-degenerate continuous coherency preserver.

Basic strategy: prove that $\phi$ preserves adjacency;

Basic technique: analyze the action on light cones.

Let $\phi : V \to V$ be a non-degenerate continuous coherency preserver.

Basic strategy: prove that $\phi$ preserves adjacency;

Basic technique: analyze the action on light cones.

# Step 1

$\phi$ never constant on a coherent line.

*Consequence:*
$\phi$ maps any coherent line into a *unique* coherent line.

# Step 1

$\phi$ never constant on a coherent line.

*Consequence:*
$\phi$ maps any coherent line into a *unique* coherent line.

# Step 2

An equivalence relation of $\mathcal{C}(0) \setminus \{0\}$:

$$a \sim b \underset{\text{def}}{\Leftrightarrow} \exists \lambda \in \mathbb{R}^* : \ b = \lambda a$$

Set

$$\mathcal{Q} := (\mathcal{C}(0) \setminus \{0\}) / \sim$$

*Notes:*

- $\mathcal{Q}$ is a projective quadric, homeomorphic to the 2-sphere;
- for each $a \in V$, we have a natural bijection from $\mathcal{Q}$ to the set of all lines in $\mathcal{C}(a)$.

For all $a \in V$,

$$\phi\big(\mathcal{C}(a)\big) \subset \mathcal{C}\big(\phi(a)\big)$$

yields

$$\varphi_a : \mathcal{Q} \to \mathcal{Q}.$$

Then, one proves that

$$a \in V \mapsto \varphi_a \in \mathcal{C}(\mathcal{Q}, \mathcal{Q})$$

is continuous.

# Step 3

If $\varphi_a$ non constant *and* $\exists b \in \mathcal{C}(a) \setminus \{a\}$ s.t. $\phi(a) = \phi(b)$, then $\phi^{-1}\{\phi(a)\}$ has non-empty interior.

Same conclusion if $\varphi_a$ nonconstant and non-injective.

**Definition**

State $a$ called **generic** if $\phi^{-1}\{\phi(a)\}$ has empty interior.

*Note:* there are generic points.

# Step 3

If $\varphi_a$ non constant *and* $\exists b \in \mathcal{C}(a) \setminus \{a\}$ s.t. $\phi(a) = \phi(b)$, then $\phi^{-1}\{\phi(a)\}$ has non-empty interior.

Same conclusion if $\varphi_a$ nonconstant and non-injective.

## Definition

State $a$ called **generic** if $\phi^{-1}\{\phi(a)\}$ has empty interior.

*Note:* there are generic points.

# Step 3

If $\varphi_a$ non constant *and* $\exists b \in \mathcal{C}(a) \setminus \{a\}$ s.t. $\phi(a) = \phi(b)$, then $\phi^{-1}\{\phi(a)\}$ has non-empty interior.

Same conclusion if $\varphi_a$ nonconstant and non-injective.

## Definition

State $a$ called **generic** if $\phi^{-1}\{\phi(a)\}$ has empty interior.

*Note:* there are generic points.

# Reformulation of Step 3

If $a$ generic and $\varphi_a$ nonconstant then:

- $\varphi_a$ injective;
- For each $b$ adjacent to $a$, $\phi(b)$ adjacent to $\phi(a)$.

# Step 4

There exists $c \in V$ generic s.t. $\varphi_c$ nonconstant.

Consequences:

- $\varphi_c$ injective (Step 3);
- $\varphi_c$ a homeomorphism of $\mathcal{Q}$ (invariance of domain theorem + compactness and connectedness of $\mathcal{Q}$);
- $\varphi_b$ nonconstant for all $b \in \mathcal{Q}$ (homotopy theory of spheres).

# Step 4

There exists $c \in V$ generic s.t. $\varphi_c$ nonconstant.

Consequences:

- $\varphi_c$ injective (Step 3);
- $\varphi_c$ a homeomorphism of $\mathcal{Q}$ (invariance of domain theorem + compactness and connectedness of $\mathcal{Q}$);
- $\varphi_b$ nonconstant for all $b \in \mathcal{Q}$ (homotopy theory of spheres).

# Step 4

There exists $c \in V$ generic s.t. $\varphi_c$ nonconstant.

Consequences:

- $\varphi_c$ injective (Step 3);
- $\varphi_c$ a homeomorphism of $\mathcal{Q}$ (invariance of domain theorem + compactness and connectedness of $\mathcal{Q}$);
- $\varphi_b$ nonconstant for all $b \in \mathcal{Q}$ (homotopy theory of spheres).

# Step 4

There exists $c \in V$ generic s.t. $\varphi_c$ nonconstant.

Consequences:

- $\varphi_c$ injective (Step 3);
- $\varphi_c$ a homeomorphism of $\mathcal{Q}$ (invariance of domain theorem + compactness and connectedness of $\mathcal{Q}$);
- $\varphi_b$ nonconstant for all $b \in \mathcal{Q}$ (homotopy theory of spheres).

# Conclusion of the proof

For all adjacent $a, b$ with $a$ generic, $\phi(a)$ and $\phi(b)$ adjacent (see Step 3).

Then, every point is generic and one concludes.

# Conclusion of the proof

For all adjacent $a, b$ with $a$ generic, $\phi(a)$ and $\phi(b)$ adjacent (see Step 3).

Then, every point is generic and one concludes.

# Possible further research

- Extend the result to continuous coherency preservers $\mathcal{H}_n \to \mathcal{H}_n$.
- Consider a more general regular real quadratic space $(V, q)$.

# Possible further research

- Extend the result to continuous coherency preservers $\mathcal{H}_n \to \mathcal{H}_n$.
- Consider a more general regular real quadratic space $(V, q)$.